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A Pair of Curves of the Fourth Degree and their Application in the Theory of Quadrics.

By H. B. Newson.

The curves in question are derived from the ellipsoid as follows: Let an ellipsoid be intersected by a series of planes forming an axial pencil, the axis of the pencil being perpendicular to the plane of the greatest and medium axes of the ellipsoid. The locus of the foci of the series of plane sections of the ellipsoid is a plane curve which I shall call throughout the Locus of Foci. The directrices of the same conic sections form a cylindrical surface whose section by the plane of the locus of the foci I shall call the Locus of Directrices. The equations of the curves are obtained as follows:

Let a, b, c be the axes of the ellipsoid in order of magnitude, and let its equation be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The plane $z = 0$ cuts from the ellipsoid a conic called the given conic, whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and from the axial pencil a flat pencil whose vertex may be represented by (m, n). Each ray of this pencil is the transverse axis of the conic cut from the ellipsoid by the plane corresponding to the ray. The locus of centres of these sections is plainly a conic similar to the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and similarly placed, passing through the point (m, n) and the centre of the given conic. Transform the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to new axes parallel to the old and having the point (m, n) for origin, and then to polar coordinates. The equation of the given conic finally becomes

$$\left.\begin{aligned} r = & \frac{m\,(1 - e^2)\cos\theta + n\sin\theta}{1 - e^2\cos^2\theta} \\ & \pm \frac{\sqrt{(b^2 - m^2(1 - e^2) - n^2)(1 - e^2\cos\theta) + (m\,(1 - e^2)\cos\theta + n\sin\theta)^2}}{1 - e^2\cos^2\theta}\,. \end{aligned}\right\} \quad (1)$$

The rational part of this value of r denotes the radius vector of the locus of centres; and the radical part, the semi-transverse axes of the series of conic sections. If A represents the semi-transverse axis of a conic and E the eccentricity, then AE is the distance from the focus to the centre. But E in this case is $=\sqrt{e_2^2\cos^2\theta + e_3^2\sin^2\theta}$ (see Annals of Mathematics, Vol. V, page 5). Hence if we multiply the radical part of (1) by the value of E, the result is the radius vector of the locus of foci. Thus the polar equation of the curve sought is

$$\left.\begin{aligned} R &= \frac{m(1-e_1^2)\cos\theta + n\sin\theta}{1-e_1^2\cos^2\theta} \\ &\pm \frac{\sqrt{\{(b^2-m^2(1-e_1^2)-n^2)(1-e_1^2\cos^2\theta) + (m(1-e_1^2)\cos\theta + n\sin\theta)^2\}(e_2^2\cos^2\theta + e_3^2\sin^2\theta)}}{1-e_1^2\cos^2\theta} \end{aligned}\right\} \quad (2)$$

Transforming this back to rectangular coordinates by writing $\sqrt{x^2+y^2}$ for R, $\frac{x}{\sqrt{x^2+y^2}}$ for $\cos\theta$, and $\frac{y}{\sqrt{x^2+y^2}}$ for $\sin\theta$, and remembering that $e_1^2 = \frac{a^2-b^2}{a^2}$, $e_2^2 = \frac{a^2-c^2}{a^2}$, $e_3^2 = \frac{b^2-c^2}{b^2}$, we get

$$\left.\begin{aligned} &\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2mx}{a^2} - \frac{2ny}{b^2} + \frac{m^2}{a^2} + \frac{n^2}{b^2} - 1\right)(x^2+y^2) \\ &\qquad + c^2\left\{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - \left(\frac{my-nx}{ab}\right)^2\right\} = 0. \end{aligned}\right\} \quad (3)$$

Hence the locus of foci is a curve of the fourth degree.

The equation of the locus of directrices is obtained in similar manner. Remembering that the distance from the centre to directrix of a conic is $\frac{A}{E}$, we have

$$\left.\begin{aligned} R &= \frac{m(1-e_1^2)\cos\theta + n\sin\theta}{1-e_1^2\cos^2\theta} \\ &\pm \frac{\sqrt{(b^2-m^2(1-e_1^2)-n^2)(1-e_1^2\cos^2\theta) + (m(1-e_1^2)\cos\theta + n\sin\theta)^2}}{(1-e_1^2\cos^2\theta)\sqrt{e_2^2\cos^2\theta + e_3^2\sin^2\theta}}. \end{aligned}\right\} \quad (4)$$

In rectangular coordinates this becomes

$$\left.\begin{aligned} &\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2mx}{a^2} - \frac{2ny}{b^2} + \frac{m^2}{a^2} + \frac{n^2}{b^2} - 1\right)(x^2+y^2) \\ &\qquad = c^2\left\{\left(\frac{mx}{a^2} + \frac{ny}{b^2}\right)^2 + \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2mx}{a^2} - \frac{2ny}{b^2}\right)\right\} \end{aligned}\right\} \quad (5)$$

Hence the locus of directrices is also a curve of the fourth degree.

The point (m, n) is evidently a double point on each curve. The curves have double contacts with each other and with the given conic. The form of each curve varies greatly with the position of the point (m, n), which may be any point in the plane. We wish now to find the envelope of the loci of foci when the point (m, n) varies. Writing $(x + m)$ for x and $(y + n)$ for y in (3), the centre of the given conic becomes the origin. Equation (3) becomes

$$\left.\begin{aligned} &\{(x+m)^2 + (y+n)^2\}\left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right\} \\ &\qquad + c^2\left\{\frac{(x+m)^2}{a^2} + \frac{(y+n)^2}{b^2} - \frac{(my-nx)^2}{a^2b^2}\right\} = 0. \end{aligned}\right\} \quad (6)$$

Arranging this according to powers of m and equating to zero the discriminant with respect to m, we have after reduction :

$$\left[(y+n)^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \frac{c^2}{a^2b^2}(x^2 + y^2 - a^2 - b^2 + c^2)\right\}\right] = 0. \quad (7)$$

The squared factor is the locus of nodes, the given conic and the focal conic, $\frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} = 1$, are envelopes.

By this method the equation of the focal conic is obtained without a previous knowledge of its properties, and might serve as a starting point for the whole theory of focal conics.

In exactly the same manner it can be shown that the envelopes of the loci of directrices are the given conic and the dirigent* conic (which is the polar reciprocal of the focal conic with respect to the given conic). With the origin at (m, n), the maximum and minimum radii vectores of the locus of foci are those to the points of contact with the focal conic.

It is not intended in this paper to discuss fully the properties of these interesting curves, but rather to make use of them in the investigation of certain properties of quadrics.

If the point (m, n) be at infinity, the locus of foci reduces to a conic. This is a well-known result. It may be shown as follows: Let $m = \infty$, $n = \infty$,

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* So called by MacCullagh, see Proceedings Royal Irish Academy.

$\frac{n}{m} = \tan\theta$ in equation (6). Whence

$$\left.\begin{aligned}&\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\\&=\frac{c^2}{a^2b^2(1+\tan^2\theta)}\left(y-x\tan\theta+\sqrt{a^2\tan^2\theta+b^2}\right)\left(y-x\tan\theta+\sqrt{a^2\tan^2\theta+b^2}\right)\end{aligned}\right\}\tag{8}$$

which is the equation of a conic passing through the points where the two lines on the right cut the given conic. But these lines are parallel tangents to the given conic, and hence the two conics have double contact, the chord of contact being the diameter of the given conic conjugate to the direction of the point (m, n).

But equation (8) may also be written in the form

$$\left.\begin{aligned}&(1+\tan^2\theta)\left(\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}-1\right)\\&=\frac{-c^2}{(a^2-c^2)(b^2-c^2)}\left(x+y\tan\theta+\sqrt{(a^2-c^2)+(b^2-c^2)\tan^2\theta}\right)\left(x+y\tan\theta-\sqrt{(a^2-c^2)+(b^2-c^2)\tan^2\theta}\right)\end{aligned}\right\}\tag{9}$$

which shows that the locus of foci has double contact also with the focal conic, for the two lines on the right-hand side are parallel tangents to the focal conic. These parallel tangents are perpendicular to the direction of (m, n), and hence the chord of contact is the diameter of the focal conic conjugate to the direction perpendicular to the direction of (m, n).

When the point (m, n) is at infinity, the locus of directrices is also a conic, a result which I think is new. Transforming equation (5) to the centre of the given conic as origin, it becomes

$$\left.\begin{aligned}&\{(x+m)^2+(y+n)^2\}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\\&\qquad=c^2\left\{\left(\frac{mx}{a^2}+\frac{ny}{b^2}\right)^2+\left(\frac{x^3}{a^2}+\frac{y^3}{b^2}\right)\left(\frac{x^3}{a^2}+\frac{y^3}{b^2}-\frac{2mx}{a^2}-\frac{2ny}{b^2}\right)\right\}\end{aligned}\right\}\tag{10}$$

Making as before $m=\infty$, $n=\infty$, and $\frac{n}{m}=\tan\theta$, this becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=\frac{c^3}{1+\tan^2\theta}\left(\frac{x}{a^2}+\frac{y\tan\theta}{b^2}\right)^2,\tag{11}$$

which is a conic having double contact with the given conic and with the locus of foci (which is now a conic), the chord of contact being the common diameter to all three conics.

Equation (10) can also be written in the form

$$\left\{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)\right\}=-\frac{c^2}{1+\tan^2\theta}\left(\frac{x\tan\theta}{a^2}-\frac{y}{b^2}\right)^2. \qquad (12)$$

The left-hand side of (12) is the equation of the dirigent conic; hence, the locus of directrices has double contact with the dirigent conic, the chord of contact being coincident with the chord of contact of the locus of foci and the focal conic.

It is easily seen that the general equation and properties of these two curves, determined for the case of the ellipsoid, are also true when the quadric is a paraboloid or an hyperboloid.

If $b = c$ in the equation of the ellipsoid, it becomes one of revolution around the axis of x. Many of the results already obtained are thereby greatly simplified. The focal conic degenerates into a right line joining the foci of a given conic. Since the loci of foci envelope the focal conic, it follows that the locus of foci always passes through the foci of the given conic, whatever may be the position of the point (m, n). This may also be shown analytically by putting $b = c$ and $y = 0$ in equation (6) and solving for x. We find $x = \pm ae$, a result independent of m and n.

Let the point (m, n) coincide with one of the foci of the given conic. Making $b = c$, $m = ae$ and $n = 0$ in equation (3), we have

$$(x^2+y^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2aex}{a^2}\right)=0. \qquad (13)$$

Thus the locus of foci degenerates into a point circle at the origin and a conic. The equation of this conic referred to its own centre is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\varepsilon^2$; hence this conic is similar and concentric to the given one and has its vertices at the foci of the latter. The meaning of the point circle is that every section through the focus has that focus for a focus. It is clear that since we are here dealing with a surface of revolution, every section by a plane passing through the axis of x is a principal section. In each such section the locus of foci is a conic as above, and the assemblage of all these conics forms a surface of revolution similar and concentric to the given one, having its vertices at the foci of the latter.

The first part of this result is well known, and was first given by M. Chasles in the Mémoires de l'Académie de Bruxelles, Vol. V, page 33. The latter part seems not so well known, but I find it given by M. Olivier in Correspondance Mathématique et Physique, Vol. V, page 391.

In the case of quadrics of revolution, the locus of directrices also breaks up when the point (m, n) coincides with one of the foci of the given conic. Let $b = c$, $m = ae$, $n = 0$ in equation (5), which then becomes

$$\left\{ex\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-(x^2+y^2)\left(\frac{1+e^2}{a}\right)\right\}\left\{x+\frac{b^2}{ae}\right\}=0. \qquad (14)$$

Thus the locus of directrices falls apart into a right line and a curve of the third degree. The right line is the directrix of the given conic corresponding to the focus on which the point (m, n) falls. This may be generalized for any principal plane of the quadric, and we have then the second part of Chasles' theorem concerning sections through a focus, viz. that one directrix of such a section lies in the polar plane of the focus through which the cutting planes pass. *The envelope of the other directrices is a surface of revolution of the third degree,* which has the plane $x = 0$ for an asymptote. This result is probably new.

Again let the point (m, n) be at the vertex of the given conic. Putting $b = c$, $m = a$, $n = 0$ in equation (5) we have

$$x^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2ax}{a^2}-\frac{b^2}{a^2-b^2}\right)=0. \qquad (15)$$

Hence the locus of directrices falls apart into a right line and a conic. The equation of this conic transformed to its own centre is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{e^2}$. Hence it is similar and concentric to the given conic and has its foci at the vertices of the latter. The tangent plane at the vertex cuts from the quadric a point circle whose directrices are indeterminate. The case is satisfied by the line $x = 0$. As before these results may be generalized for all principal sections of the quadric. I shall now bring together and express in a convenient form the theorems of Chasles and Olivier and the results just developed.

Given two quadrics of revolution similar and concentric such that the foci of the one coincide with the vertices of the other. Any cutting plane passing through the vertex of the inner surface cuts from the outer surface a conic both of whose foci are on the inner surface, but one of which is always at the vertex of the inner surface through which the cutting plane passes. One directrix of the same conic is always in a fixed plane, the directrix plane of the outer surface. The other directrix is always tangent to a surface of revolution of the third degree. The same cutting plane cuts from the inner surface a conic both of whose foci are on a surface of revolution of the fourth degree and both of whose directrices are tangent to the outer surface.

When the given surface of the second degree is a cone, the section by the plane xy is a pair of intersecting lines. The only case of special interest is when the given point (m, n) falls on one of these lines. The locus of foci and locus of directrices each break up into the right line on which the point falls and a curve of the third degree. The locus of foci in this case once attracted considerable attention and was studied in detail by a group of Belgian mathematicians.

The curve was first studied by Quetelet in a paper entitled *Dissertatio de quibusdam locis geomet. nec non de Curvâ focali:* Gandavi, 1819. This paper I have not seen. In the Journal of the Royal Academy of Brussels I find the following entry for December 6, 1819 :

"M. le Commandeur de Niewport a présenté au nom de M. Quetelet, professeur de mathématique à l'athénée de Bruxelles, un mémoire sur quelques nouvelles propriétés de la focale et sur la même courbe focale. * * * * * L'Académie avait chargé M. Quetelet de refondre ses deux mémoires en un seul; mais l'auteur ayant appris que M. Dandelin s'occupait d'un écrit sur le même sujet a cru renouncer à son travail, d'autant plus que celui de son ami ne laissait rien à desirer sur ce point."

Dandelin's paper is published in the Mémoires de l'Académie de Bruxelles, Vol. II. It deals only with the case of the right circular cone. A memoir by Van Rees in Quetelet's Correspondance Mathématique et Physique, Vol. VI, page 361, treats of the curve in both the principal sections of the elliptic cone.

In case of the elliptic cone the curve consists of two branches, each of which is tangent to a focal line of the cone. One branch touches the given line at the point (m, n) and the other branch has the other given line for an asymptote. If the cone is right circular, the locus of foci has a double point and is the "focale a nœud" of Quetelet recently discussed by Morley (*American Journal,* XI, page 307).

The locus of directrices in the case of the cone is a curve of the third degree whenever the locus of foci is of the third degree. The Belgian mathematicians who studied the properties of the locus of foci do not seem to have noticed the locus directrices. I add without proof a number of theorems on the focal properties of series of quadrics which may readily be proved by means of the methods and formulas already developed.

Given a series coaxial and concentric hyperboloids of revolution of one nappe all having the same circle of the gorge; if the axis of an axial pencil of planes is tangent to the circle of the gorge, the locus of foci of parabolic sections is a circle; the locus of directrices of the same sections is also a circle.

If the axis of an axial pencil of planes is perpendicular to the axial plane of a parabolic cylinder and tangent to the cylinder, the locus of foci of parabolic sections is a circle, so is also the locus of directrices.

Given a series of right circular cones on the same base, if the axis of an axial pencil of planes is tangent to the base circle, the locus of foci of parabolic sections is a circle; so is also the locus of directrices.

The eccentricity of the section of a quadric of revolution by a plane passing through a focus and containing any tangent to the central circular section is equal to the square of the eccentricity of the quadric.

If the axis of an axial pencil of planes is tangent to the base circle of a right circular cone, the distance from the centre to the foci of any section is equal to the distance from the centre of the section to the centre of the base circle. Given the same conditions as above, the perpendicular from the centre of the base circle on the parabolic section passes through the focus of the parabola.